# An optimum PML for scattering problems in time-domain

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***Abstract* — The Perfectly Matched Layer (PML) is widely used for unbounded problems. However its performances depend critically on an absorption coefficient. This coefficient is generally tuned by using case-dependent optimization procedures. In this paper we will present some efficient profiles of the coefficient that overcome every tuning. These profiles and others will be compared by using a scattering benchmark solved with the Discontinuous Galerkin method.**

## I. INTRODUCTION

A major challenge for the numerical simulation of wave-like problems on unbounded domain, such as scattering problems, is to truncate the computational domain without altering too much the original solution. Different techniques have been developped to simulate the artificial boundary produced by the truncation. The Perfectly Matched Layer (PML) technique, introduced by Berenger [1], is one of them. It is now developped and widely used in both frequency and time domains.

The idea is to surround the truncated computational domain with a specific layer. Inside it, the physical medium is adapted in such a way that outgoing waves are progressively damped and seem to leave the domain without reflection. To achieve this, the differential equations of the problem are transformed and an absorption coefficient $\sigma$ governing the damping is introduced. In time domain the definition of one or more additional fields is needed.

This paper deals with the choice of the coefficient $\sigma$ that is still tricky when numerical methods are used. We present hyperbolic functions for $\sigma$ that provide good results without any tuning, unlike the commonly used absorption coefficients that require case-dependent optimization procedures.

A PML and a Discontinuous Galerkin (DG) method are introduced in section 2 for a scattering problem in time domain. Hyperbolic functions for $\sigma$ are then presented and compared with case-dependent-optimized polynomial profiles in section 3. A scattering application exemple is finally present in section 4.

## II. MATHEMATICAL AND NUMERICAL FRAMEWORK

Consider the two-dimensional scattering of a perfectly conducting obstacle in an unbounded area. The two-dimensional computational domain $\Omega$ is truncated by a circular artificial boundary $\Gamma$ of radius $R$, and is surrounded by a PML $\Omega_{pml}$ with a uniform thickness $\delta$. The incident fields are prescribed using a total/scattered fields transition at $\Gamma$. A scattered fields formulation is then used in $\Omega_{pml}$ while a total fields one is used in $\Omega$.

In the transverse magnetic case the total, scattered and incident electric fields have only one component along $\mathbf{e}_z$, *i.e.* $\mathbf{E}^s = [0, 0, E_z^s]$ for the scattered field, while the magnetic ones are perpendicular to $\mathbf{e}_z$, *i.e.* $\mathbf{H}^s = [H_x^s, H_y^s, 0]$. The non-null components of these fields are governed by the Maxwell's equations in $\Omega$. In the PML $\Omega_{pml}$, the scattered fields are governed by the modified cartesian equations

$$\varepsilon \frac{\partial \mathbf{E}^s}{\partial t} = \nabla \times \mathbf{H}^s - \varepsilon(\sigma - \bar{\sigma})\mathbf{Q} - \varepsilon\bar{\sigma}\mathbf{E}^s, \quad (1)$$

$$\mu \frac{\partial \mathbf{H}^s}{\partial t} = \nabla \times \mathbf{E}^s - \mu(\sigma - \bar{\sigma})(\mathbf{H}^s \cdot \mathbf{e}_r)\mathbf{e}_r - \mu\bar{\sigma}\mathbf{H}^s, \quad (2)$$

where $\sigma(r)$ is the absorption coefficient, $\bar{\sigma}(r)$ is

$$\bar{\sigma}(r) = \frac{1}{r}\int_R^r \sigma(r')\, dr', \quad (3)$$

$\mathbf{e}_r$ is the radial unit vector and $r = \|\mathbf{x}\|$ is the radial coordinate. The additional vector field introduced in $(1)$ has only one component along $\mathbf{e}_z$, *i.e.* $\mathbf{Q} = [0, 0, Q_z]$, and is governed by

$$\varepsilon \frac{\partial \mathbf{Q}}{\partial t} = [\mathbf{e}_r(\mathbf{e}_r \cdot \nabla)] \times \mathbf{H}^s - \varepsilon\sigma\mathbf{Q}. \quad (4)$$

These equations are equivalent to those obtained by Collino and Monk in [4] for the transverse electric case.

The absorption coefficient $\sigma(r)$ is assumed to be positive in $\Omega_{pml}$. If it is singular at the outer boundary of the PML, *i.e.* at $r = R + \delta$, the solution must satisfy a homogeneous Dirichlet condition on one of the fields to have a well-posed problem, otherwise a Neumann condition or even a radiation condition could be implemented to improve the efficiency of the layer. In this paper a perfect-electric-conductor (PEC) condition is always prescribed.

The numerical solution is computed using a DG method with an unstructured grid of triangular cells [5]. The weak forms of the equations are integrated on each cell and numerical fluxes are defined between adjacent cells. Upwind fluxes are used for the two first equations. The numerical flux of the additionnal equation $(4)$ is obtained by multiplying $\mathbf{e}_r$ by the projection of the numerical flux of $(1)$ in the direction $\mathbf{e}_r$. The integrals on each cell and each interface are then computed using a full Gaussian quadrature procedure. Finally, all the results are obtained with linear basis functions and a Crank-Nicolson time stepping scheme.

## III. OPTIMIZING THE PML

The large succes of the PML technique is mainly due to the perfectly matching property of the interface between the PML medium and the physical one in the continuous context. It ensures the perfect transmission of waves from one medium to the other one without reflection. Moreover, for a PML with a finite thickness $\delta$, any reflection is avoided if $\sigma$ is singular at the outer boundary of the PML (see [2] or [3]). However this condition does not apply to the discrete contexts and the perfectly matching is not ensure yet.

When numerical methods are used the properties of the PML depend crutially on the coefficient $\sigma$, the thickness $\delta$

and the discrete mesh. The continuous properties of the PML could be nearly recover by tacking a large thickness $\delta$ or by using a very fine mesh. Nevertheless the computationnal cost inscreases.

The coefficient $\sigma$ is the most crucial parameter. Indeed the performance of the PML decreases dramatically when the damping of the solution induced by large values of $\sigma$ cannot be captured by the discretization grid [6]. Therefore $\sigma$ must be chosen in such a way to introduce enough damping of outgoing waves without inducing a too sharp decrease of the fields in the PML. A polynomial profile of $\sigma$ allows such a progressive damping:

$$\sigma_p(r) = \alpha[(r-R)/\delta]^n, \tag{5}$$

with $n > 0$ and where $\alpha$ is the value of the absorption coefficient at the outer side of the layer. This profile is widely used. However no general rule exists to choose the parameters $\alpha$ and $n$. Therefore some expensive and case-dependent optimization procedures are used to tune them. As an alternative, hyperbolic profiles have been proposed by Bermúdez et al. [2]:

$$\sigma_h(r) = \alpha/[\delta - (r-R)]^n, \tag{6}$$

$$\sigma_{sh}(r) = \alpha/[\delta - (r-R)]^n - \alpha/\delta^n, \tag{7}$$

with again two free dimensionless parameters: $\alpha$ and $n$. Fortunately these two profiles do not require any tuning with $n = 1$. Bermudez et al. [2] shows in a continuous finite elements context with a straight cartesian PML that the solution is indeed close to the optimum when $\alpha$ is close to the propagation velocity of the medium. We extend this result to the finite differences and the DG contexts in [6] and [7]. Note that, in practice, we replace $\delta$ by $1.01\delta$ in (6) and (7) to avoid the numerical integration of the singular point at the end of the layer when DG is used.

## IV. NUMERICAL SIMULATION

We show in this section that the previous results are still valid for a more realistic scattering benchmark.

Consider the scattering of the cavity described in [8] by an incident Gaussian pulse. The truncated computational domain is a disk with a radius of $0.5m$, and surrounded by a PML with the thickness $\delta = 0.12m$ (Figure 1). The Gaussian pulse comes from the left side of the domain and propagates to the right during $4\ 10^{-9}s$, with the time step $\Delta t = 5\ 10^{-11}s$. At the end of the simulation, we compute the $L2-$norm of the error on the fields,

$$\sqrt{\frac{\int_\Omega \frac{1}{2}\left(\varepsilon_0 \left\|\mathbf{E}-\mathbf{E}^{ref}\right\|^2 + \mu_0 \left\|\mathbf{H}-\mathbf{H}^{ref}\right\|^2\right) d\Omega}{\int_\Omega \frac{1}{2}\left(\varepsilon_0 \left\|\mathbf{E}^{ref}\right\|^2 + \mu_0 \left\|\mathbf{H}^{ref}\right\|^2\right) d\Omega}}, \tag{8}$$

where $\mathbf{E}^{ref}$ and $\mathbf{H}^{ref}$ are the reference fields computed with a larger computational domain.

Figure 2 shows the final error (8) as a function of $\alpha$ for the different $\sigma(x)$. We show that a minimum value of these curves are slightly lower to $\alpha = 1/\sqrt{\mu_0\varepsilon_0}$ for the functions $\sigma_h$ and $\sigma_{sh}$. The use of $\sigma_{sh}$ with $\alpha = 1/\sqrt{\mu_0\varepsilon_0}$ gives an error closed to those obtained with other functions at the best cases.

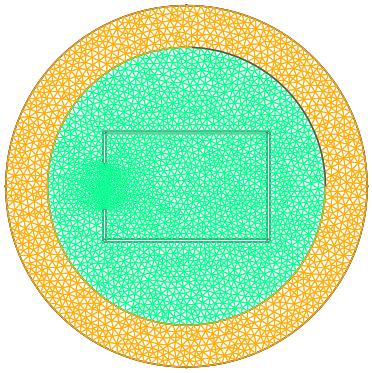

Fig. 1. Geometry and mesh of the benchmark.

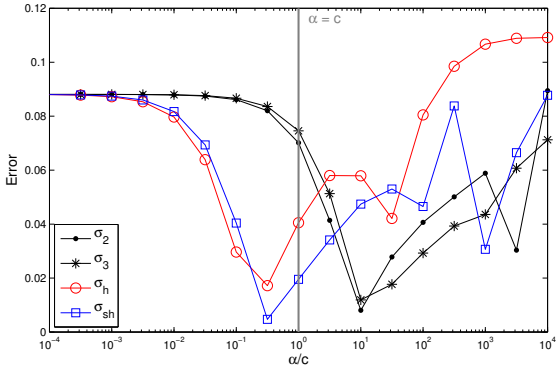


Fig. 2. Final error of the fields for different $\alpha$ with the different absorption coefficients. The vertical line indicates $\alpha = c$, where $c$ is the propagation velocity $1/\sqrt{\mu_0\varepsilon_0}$.

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