A PML for Convex Truncated Domains in Time-Dependent Acoustics with a Discontinuous Galerkin Finite Element Discretization

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Abstract

A new perfectly matched layer (PML) is proposed for convex truncated domains in the context of time-dependent acoustics. With this formulation, the size of the computational domain can be reduced when classical shapes of truncated domains are less appropriate. A numerical discretization based on the discontinuous Galerkin method is then described and validated. An example of realistic three-dimensional application is finally proposed.

Introduction

Perfectly matched layers (PMLs) are used to truncate the computational domain of wave-like problems defined on unbounded spatial domains. Various versions have been proposed in the classical curvilinear coordinate systems to deal with planar, cylindrical or spherical truncations. In [4], the PML is extended to general convex truncated domains in a time-harmonic electromagnetic context. A general convex truncation permits the decrease of the computational cost by diminishing the size of the computational domain when the classical shapes of truncation are less appropriate.

In this paper, we derive a new PML for convex truncations in the time domain. In Section 1, the PML equations are written for three-dimensional problems. In Section 2, we propose a numerical discretization using the discontinuous Galerkin finite element method. The obtained numerical scheme is validated using a three-dimensional reference benchmark. A realistic application is finally presented in Section 3.

1 Governing equations in the PML

We consider the time-evolution of the pressure $p(\boldsymbol{x},t)$ and the velocity $\boldsymbol{u}(\boldsymbol{x},t)$ in the convex bounded domain $\Omega \in \mathbb{R}^3$, that is surrounded by a PML Ω_{pml} with a constant thickness δ. The external boundary Γ of Ω is assumed to be sufficiently smooth.

Employing the strategy proposed in [2], the governing equations in the PML are built by using a complex stretching of spatial coordinates in the frequency domain. For this purpose, the time-harmonic equations are firstly written in a convenient curvilinear coordinate system. Following [4], we consider the system associated to the orthonormal local basis $(\boldsymbol{n}, \boldsymbol{t}_\varphi, \boldsymbol{t}_\theta)$ where, for a point P of Ω_{pml}, the unit vectors $\boldsymbol{n}$, $\boldsymbol{t}_\varphi$ and $\boldsymbol{t}_\theta$ are the external normal and the two principal directions of the surface Γ at the closest point P^Γ of Γ to P. The curvilinear coordinate r associated to the direction $\boldsymbol{n}$, that corresponds to the distance between P and P^Γ, is then stretched using

$$r \quad \rightarrow \quad \tilde{r} = r - \tfrac{1}{i\omega} \textstyle\int_0^r \sigma(r')dr',$$

where $\sigma(r)$ is the absorption function. Following [3], we use the hyperbolic function $\sigma(r) = cr/(\delta - t)$, that does not require any optimization. Time-dependent cartesian equations are finally obtained by performing an inverse Fourier transform in time, by defining additional differential equations and by moving back to the cartesian coordinate system.

In both Ω and Ω_{pml}, the fields are then governed by the equations

$$\partial_t p + \rho c^2 \, \nabla \cdot \boldsymbol{u} = s_p, \tag{1}$$

$$\partial_t \boldsymbol{u} + \rho^{-1} \, \nabla p = \boldsymbol{s}_{\boldsymbol{u}}, \tag{2}$$

where ρ and c are positive constants. In Ω, the classical equations are recovered considering the source terms s_p and $\boldsymbol{s}_{\boldsymbol{u}}$ equal to zero while, in Ω_{pml}, these terms are

$$\begin{aligned} s_p &= -\sigma p_n - \bar{\kappa}_\varphi \bar{\sigma} p_\varphi - \bar{\kappa}_\theta \bar{\sigma}(p - p_n - p_\varphi), \\ \boldsymbol{s}_{\boldsymbol{u}} &= -\sigma \boldsymbol{n}(\boldsymbol{n} \cdot \boldsymbol{u}) - \bar{\kappa}_\varphi \bar{\sigma} \boldsymbol{t}_\varphi(\boldsymbol{t}_\varphi \cdot \boldsymbol{u}) - \bar{\kappa}_\theta \bar{\sigma} \boldsymbol{t}_\theta(\boldsymbol{t}_\theta \cdot \boldsymbol{u}), \end{aligned} \tag{3}$$

with $\bar{\kappa}_\varphi = (\kappa_\varphi^{-1} + r)^{-1}$, $\bar{\kappa}_\theta = (\kappa_\theta^{-1} + r)^{-1}$, $\bar{\sigma} = \int_0^r \sigma(r')dr'$, where κ_φ and κ_θ are the main curvatures of Γ at P^Γ. Finally, the two additionnal fields p_n and p_φ introduced in the equation (3) are governed by

$$\partial_t p_n + \rho c^2 \, [\boldsymbol{n}(\boldsymbol{n} \cdot \nabla)] \cdot \boldsymbol{u} = -\sigma p_n, \tag{4}$$

$$\partial_t p_\varphi + \rho c^2 \, [\boldsymbol{t}_\varphi(\boldsymbol{t}_\varphi \cdot \nabla)] \cdot \boldsymbol{u} = -\bar{\kappa}_\varphi \bar{\sigma} p_\varphi. \tag{5}$$

2 Numerical scheme

Discontinuous Galerkin method

The equations above are solved using a nodal DG finite element scheme [1] with a mesh made of tetrahedra. The scheme is built by considering the conservative form of the governing equations and by multiplying them by test functions. Integrating the resulting equations over a cell and using integration by part leads to the weak form. The numerical fluxes used in the interface terms of the first two equations (1) and (2) are defined using a Riemann solver, while Lax-Friedrichs fluxes are considered for the two additional equations (4) and (5). Each scalar field and each cartesian component of $\boldsymbol{u}$ is approximated by a first-order Lagrange polynomial. The time-stepping is made with the fourth-order Runge-Kutta method.

Validation

To validate the method, we consider a truncated domain shaped as an ellipsoid of revolution and surrounded by a PML of thickness $\delta = 500$ m. The lengths of the axis of the ellipsoid are 6.6 km ($x-$direction) and 2.4 km ($y-$ and $z-$directions). A Gaussian is prescribed as initial condition on p, *i.e.*

$$p(\boldsymbol{x}, 0) = e^{-\|\boldsymbol{x}-\boldsymbol{x}_0\|^2/R^2}$$

with $\boldsymbol{x}_0 = (-2.45 \text{ km}, 0, 0.4 \text{ km})$ and $R = 150$ m, while the other fields are initially equal to zero. We use $c = 1.5$ km/s and $\rho = 1$ kg/m^3. As time goes by, spherical waves are generated and reach the PML with different incidences.

During the simulation, the numerical solution is compared with the exact solution in the truncated domain Ω. Figure 1 shows the convergence of the relative mean error ξ_r defined by

$$\frac{\int_0^{t_f}\int_\Omega \left(\frac{1}{2\rho c^2}(p_{\text{ana}} - p_{\text{num}})^2 + \frac{\rho}{2}\|\boldsymbol{u}_{\text{ana}} - \boldsymbol{u}_{\text{num}}\|^2\right) d\Omega\, dt}{\int_0^{t_f}\int_\Omega \left(\frac{1}{2\rho c^2}(p_{\text{ana}})^2 + \frac{\rho}{2}\|\boldsymbol{u}_{\text{ana}}\|^2\right) d\Omega\, dt}$$

and computed for the duration $t_f = 4$ s.

3 Realistic benchmark

A submarine is added in the geometry described above. Figure 2 shows the snapshot of p at two instants of the simulation. The spherical waves are not deformed near the external boundary of the domain Ω and are damped in the PML.

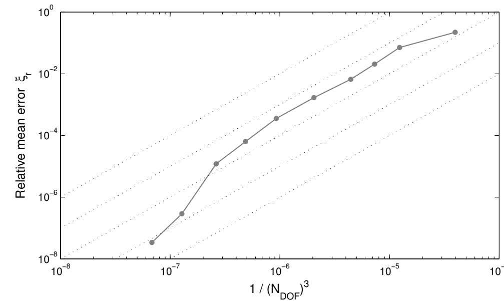

Figure 1: Convergence of the relative mean error. N_{DOF} is the total number of degrees of freedom.

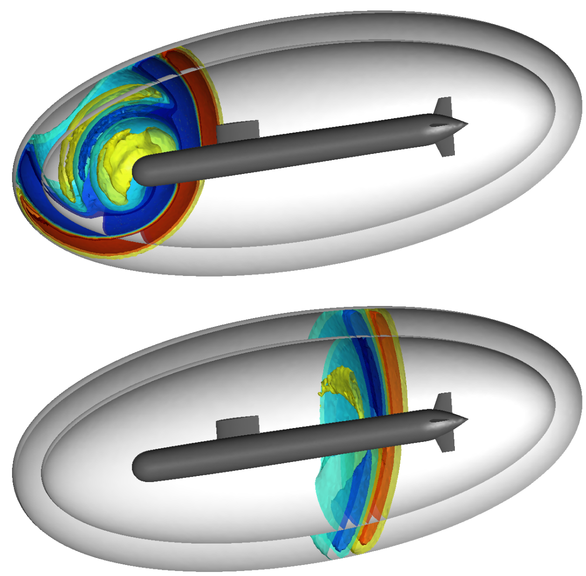

Figure 2: Iso-surfaces of p at $t = 0.8$ s *(top)* and $t = 2.4$ s *(bottom)* for the realistic benchmark.

References

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